PHYSICAL REVIEW VOLUME 129, NUMBER 1 1 JANUARY 1963

Transverse Conductivity of an Electron Gas: Zero-Frequency Limit*

M. L. GLASSER
Battelle Memorial Institute, Columbus, Ohio
(Received 3 August 1962)

The real part of the transverse conductivity of a dense electron gas at zero temperature is evaluated in perturbation theory including all diagrams which contribute in the low-frequency limit. It is found that there are no many-body corrections to the result for noninteracting electrons. The effect of a low density of randomly placed impurities is also considered. In this case, the calculation leads to the transport equation derived by Silin from the Landau theory of Fermi fluids.

I. INTRODUCTION

IN this paper, we examine the response of an idealized metal, at zero temperature, to a low-frequency transverse electromagnetic field. The emphasis is on the role of electron correlation in the region of the anomalous skin effect. This problem has been studied by Silin[1] using Landau's[2] theory of Fermi liquids. He showed that the Coulomb interaction has no effect on the surface impedance of a metal in the anomalous skin effect limit. This point was recently brought into question, however, when Stern and Falicov and Heine[3] suggested, on the basis of quasi-particle models, that the electron interactions could give rise to an observable charge renormalization. This work has now been clarified[4] and appears to be consistent with Silin's calculation.

The Landau theory on which Silin's work is based is phenomenological and, insofar as it is concerned with a coordinate and momentum-dependent distribution function, is inherently semiclassical. The purpose of this calculation is to substantiate Silin's results on the basis of a microscopic approach. To do this, we employ many-body perturbation theory starting from a noninteracting electron gas. It should be realized, however, that in presuming a convergent, resummable perturbation expansion, the present approach may be no less phenomenological than the Landau theory.

In this work, the model used by Langer[5] in his study of dc resistivity is adopted. This consists of a dense interacting electron gas neutralized by a uniform background of positive charge. In it are imbedded randomly placed, fixed point charges and the entire system is enclosed in a cubic volume which will eventually be taken as infinite. Periodic boundary conditions are used. For conciseness we shall follow Langer's analysis closely and refer the reader to his papers for details.

The picture which emerges is essentially that presented by Falicov and Heine.[6] The external field produces excitations which, as long as they occur in the vicinity of the Fermi surface, are stable and interact only weakly among themselves. Such a "quasi-particle" is, in essence, an electron surrounded by a screening hole. In the presence of impurities these entities are described by wave packets localized within a mean free path (roughly the inter-impurity distance) which move at their group velocity and carry precisely the unscreened electronic charge. For weak external fields the distribution of these excitations is described by a transport equation which reduces to the usual Boltzmann equation in the absence of interaction. (The transport equation properly applies only near the Fermi surface. For details see Heine, reference 4.)

In the following section we derive the basic formalism and express the response in terms of a conductivity. Section III concerns the case of a "free" electron gas (no impurities) and the derivation of the transport equation is presented in Sec. IV.

II. GENERAL FORMALISM

We take the interacting electron system in its ground state Ψ_0 and turn on slowly the external field represented by the vector potential

$$\mathbf{A}(\mathbf{x},t)=e^{i\mathbf{q}\cdot\mathbf{x}+i\omega t+\alpha t}\mathbf{A}(\mathbf{q},\omega)$$

(the limit $\alpha\rightarrow 0^+$ will eventually be taken). If $\Psi(t)$ denotes the state which arises adiabatically from Ψ_0, then the induced current is given by

$$\mathbf{J}^{\text{ind}}(\mathbf{x},t)=\left\langle\Psi(t)\left|\frac{e}{m}\sum_{i=1}^{N}\left(\mathbf{p}_i-\frac{e}{c}\mathbf{A}(\mathbf{x},t)\right)\delta(\mathbf{x}-\mathbf{x}_i)\right|\Psi(t)\right\rangle,$$

where N is the number of electrons. To obtain the linear response we shall retain only terms to first order in the field variables.[7] To this approximation

$$\Psi(t)=e^{-iHt}\Psi_0-ie^{-iHt}\int_{-\infty}^{t}e^{iHt'}H_1(t')e^{-iHt'}\Psi_0dt',$$

* This work is based on a thesis submitted to the Carnegie Institute of Technology in partial fulfillment of the requirements for the Ph.D., and was supported in part by the National Science Foundation.

[1] V. P. Silin, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 495 (1957) [translation: Soviet Phys.—JETP **6**, 387 (1958)].

[2] L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **30**, 1058 (1956) [translation: Soviet Phys.—JETP **3**, 920 (1956)].

[3] *The Fermi Surface*, edited by W. Harrison and M. Webb (John Wiley & Sons, New York, 1960).

[4] V. Heine, Phil. Mag. **7**, 775 (1962); E. Stern, Bull. Am. Phys. Soc. **7**, 220 (1962).

[5] J. S. Langer, Phys. Rev. **120**, 714 (1960); **124**, 997 (1961); **124**, 1003 (1961). We shall refer to these as L-I, L-II, and L-III, respectively.

[6] L. M. Falicov and V. Heine, *Advances in Physics*, edited by N. F. Mott (Taylor and Francis, Ltd., London, 1961), Vol. 10, p. 57.

[7] Magnetic effects are very small in an electron gas and they will be neglected in this paper.

where H is the total electron-impurity Hamiltonian and $H_1(t)$ is the linear perturbation due to the external field.[8] Thus, we find, since the net current must vanish in the ground state,

$$J_\mu^{\mathrm{ind}}(\mathbf{x},t)=-\frac{e^2N}{mc\Omega}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}A_\mu(\mathbf{k},t)-\frac{i}{\Omega c}\sum_{\mathbf{k}\mathbf{k}'} e^{i\mathbf{k}\cdot\mathbf{x}} \times\int_{-\infty}^{t}\langle\Psi_0|[j_\mu(\mathbf{k}',t),\, j_\nu(-\mathbf{k},\,t')]|\Psi_0\rangle \times A_\nu(\mathbf{k},t')dt'.$$

$\mathbf{j}$ is the electron current operator and all quantities have been Fourier transformed with respect to the system volume Ω. After taking the four-dimensional Fourier transform and noting that $\langle\Psi_0|[j_\mu(\mathbf{q},t),j_\nu(\mathbf{q},t')]|\Psi_0\rangle$ depends only on the relative time variable $u=t-t'$, we find that the induced current has the form

$$J_\mu^{\mathrm{ind}}(\mathbf{q},\omega)=T_{\mu\nu}(\mathbf{q},\omega)A_\nu(\mathbf{q},\omega),$$

where

$$T_{\mu\nu}(\mathbf{q},\omega)=-\frac{e^2n}{mc}\delta_{\mu\nu}-\frac{i}{\Omega c} \times\int_0^\infty\langle\Psi_0|[j_\mu(\mathbf{q},u),\, j_\nu(-\mathbf{q},\,0)]|\Psi_0\rangle e^{-iu\omega}du$$

(n = electron density). Since the transverse electric field is related to the vector potential by $\mathbf{E}(\mathbf{q},\omega)=i(\omega/c)\mathbf{A}(\mathbf{q},\omega)$, the complex conductivity tensor is

$$\sigma_{\mu\nu}(\mathbf{q},\omega)=i\frac{e^2n}{m\omega}\delta_{\mu\nu}-\frac{1}{\omega\Omega} \times\int_0^\infty e^{-i\omega t}\langle\Psi_0|[j_\mu(\mathbf{q},t),\, j_\nu(-\mathbf{q},\,0)]|\Psi_0\rangle dt.$$

Because of cubic symmetry we may separate this into transverse and longitudinal parts:

$$\sigma_{\mu\nu}(\mathbf{q},\omega)=\sigma(q,\omega)[\delta_{\mu\nu}-(q_\mu q_\nu/q^2)]+\sigma'(q,\omega)(q_\mu q_\nu/q^2),$$

and we find that the transverse conductivity is

$$\sigma(q,\omega)=\tfrac{1}{2}\sum_{\mu,\nu}\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\sigma_{\mu\nu}(q,\omega)$$

$$=\frac{ie^2n}{m\omega}-\frac{i}{\omega\Omega}\int_0^\infty dt\, e^{-i\omega t} \times\mathrm{Im}\langle\Psi_0|\mathbf{j}_t(\mathbf{q},t)\cdot\mathbf{j}_t(-\mathbf{q},\,0)|\Psi_0\rangle.$$

$\mathbf{j}_t$ represents the component of the current operator transverse to $\mathbf{q}$ and has, in general, two nonvanishing components. Let

$$\mathfrak{F}(q,t)=T\langle\Psi_0|\mathbf{j}_t(\mathbf{q},t)\cdot\mathbf{j}_t(-\mathbf{q},\,0)|\Psi_0\rangle,$$

where T is the Wick chronological ordering operator. $\mathfrak{F}(q,t)$ has the form of an autocorrelation coefficient and may be expected to vary slowly and tend to zero at large t. Thus for very high frequencies the oscillations of the exponential in the integrand will cause the contribution of $\mathfrak{F}(q,t)$ to cancel so that

$$\sigma(q,\omega)\underset{\omega\to\infty}{\longrightarrow} ie^2n/m\omega.$$

This is precisely the value for noninteracting electrons.

We introduce the Fourier transform of $\mathfrak{F}(q,t)$:

$$F(q,\nu)=\frac{1}{2\pi}\lim_{\eta\to0^+}\int_{-\infty}^\infty\mathfrak{F}(q,t)e^{i\nu t-\eta|t|}dt.$$

By inserting a complete set of intermediate states $|n\rangle$, this can be put into the spectral form

$$F(q,\nu)=\frac{1}{2\pi i}\int_0^\infty d\nu'\,\rho(q,\nu')\left(\frac{1}{\nu'-\nu-i\eta}+\frac{1}{\nu'+\nu-i\eta}\right), \quad (2.1)$$

where the spectral density is

$$\rho(q,\nu)=\textstyle\sum_n|\langle\Psi_0|\mathbf{j}_t(\mathbf{q})|n\rangle|^2\delta(E_n-E_0-\nu). \quad (2.2)$$

$\rho(q,\nu)$ is just the discontinuity of $F(q,\nu)$ across the real axis in the complex ν plane. By inverting the Fourier transform, we find for positive times that

$$\mathfrak{F}(q,t)=\int_0^\infty\rho(q,\nu)e^{-i\nu t-\eta t}d\nu. \quad (2.3)$$

For nonvanishing q, the f-sum rule[9] states

$$\sum_n\frac{|\langle\Psi_0|\mathbf{j}_t(\mathbf{q})|n\rangle|^2}{E_n-E_0}=\frac{ne^2}{m}$$

(keeping in mind that $\mathbf{j}_t$ has two components). We find from (2.3) that

$$\mathrm{Im}\int_0^\infty\mathfrak{F}(q,t)dt=\int_0^\infty\frac{\rho(q,\nu)}{\nu}d\nu=-\pi\,\mathrm{Im}F(q,0).$$

Using (2.1) and (2.2), we have

$$F(q,0)=-\frac{i}{\pi}\sum_n\frac{|\langle\Psi_0|\mathbf{j}_t(\mathbf{q})|n\rangle|^2}{E_n-E_0};$$

hence

$$\mathrm{Im}\int_0^\infty\mathfrak{F}(q,t)dt=\frac{ne^2}{m}.$$

This result permits us to express the conductivity in the form

$$\sigma(q,\omega)=\frac{i}{\omega\Omega}\int_0^\infty(1-e^{-i\omega t})\,\mathrm{Im}\mathfrak{F}(q,t)dt.$$

[8] We employ units in which $\hbar=1$.

[9] P. Nozières and D. Pines, Phys. Rev. **113**, 1254 (1958).

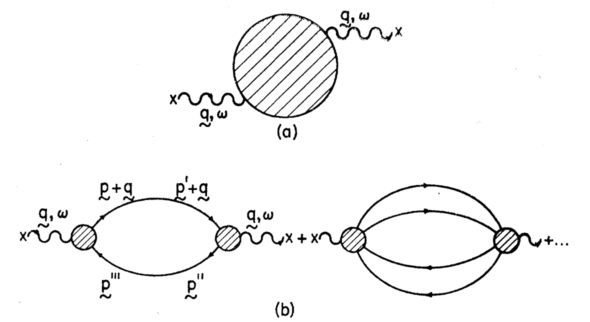

FIG. 1. (a) General form of diagrams which contribute to $\mathfrak{F}(\mathbf{q},\omega)$. (b) The reduced graph expansion associated with the real conductivity.

By making use of the spectral representation we find, for positive ω, that

$$\mathrm{Re}\sigma(q,\omega)=(\pi/2\Omega)(\rho(q,\omega)/\omega);$$

$\mathrm{Im}\sigma(q,\omega)$ may be obtained from the Kramers-Kronig relation. The quantity $\mathfrak{F}(q,t)$ is a time-ordered vacuum expectation value and may be represented by diagrams in the same way as $\mathfrak{F}(t)$ in L-I; there are some new features however. The external photon lines now carry nonvanishing momentum and frequency and contribute a factor $\mathbf{k}_t\cdot\mathbf{k}_t'$ to the matrix element rather than $\mathbf{k}\cdot\mathbf{k}'$. The elimination of unlinked clusters and disconnected diagrams proceeds just as in L-I, but in the present case improper diagrams can also occur. These are diagrams which can be disconnected by cutting a single internal photon line. It is well known, however, that such processes can have no effect on the transverse current and these diagrams will be neglected. Thus, in particular, there are no random phase approximation corrections to the conductivity for noninteracting electrons as has been pointed out by Nozières and Pines.[9] Contributions to $\mathfrak{F}(q,t)$ therefore come only from connected, proper diagrams of the form shown in Fig. 1(a). Each such diagram corresponds to an amplitude whose Fourier transform has a branch cut along the real axis in the complex frequency plane.

The conductivity depends explicitly on the quantity $\rho(q,\omega)$ which is the sum of all the discontinuities of these diagrams. The discontinuities may be evaluated and summed directly by means of the analytic techniques described in L-II. In this manner we obtain the real part of the conductivity in the form of a reduced graph expansion as illustrated in Fig. 1(b). The shaded circles represent vertex corrections giving rise to multiple quasi-particle-hole pairs in intermediate states which lie on the energy shell. (The particle-hole lines include all possible self-energy corrections and may be considered to represent quasi-particles.) It is shown in L-II that a class n reduced graph, i.e., one having n intermediate pairs, gives rise to a discontinuity proportional to ω^{2n-1}. It appears that it is also proportional to $[1/(qv_F)]^{2n-1}$, where v_F is the noninteracting electron Fermi velocity, so that the reduced graph expansion is one in odd powers of (ω/qv_F). This paper is concerned with the zero-frequency limit to which only the class 1 reduced graphs contribute. We hope to return to the finite frequency case in a future report. (It is not hard to show explicitly that many types of class n diagrams, such as the ones considered by Tzoar and Klein,[10] are negligible for $\omega\ll\omega_p$, the plasma frequency.)

We now consider only class 1 reduced diagrams for a given distribution of impurities. The sum of all possible reduced graphs of this type is obtained by inserting for each of the shaded circles in the first diagram of Fig. 1(b) the sum of all proper vertex functions (see L-II) $\mathbf{\Lambda}_\mathbf{q}(\mathbf{p},\mathbf{p}',\omega)$. The corresponding contribution to the conductivity is then

$$\begin{aligned}\sigma(q)=\lim_{\omega\to0}\mathrm{Re}\sigma(q,\omega)=\lim_{\omega\to0}\frac{\pi}{\omega\Omega}\sum_{\mathbf{p}'s}\sum_{\lambda=1}^{2}\int_\mu^\infty d\xi_1\\ \times\int_{-\infty}^{\mu}d\xi_2\Lambda_\mathbf{q}^\lambda(\mathbf{p}''',\mathbf{p},\omega)A(\mathbf{p}+\mathbf{q},\,\mathbf{p}'+\mathbf{q},\,\xi_1)\\ \times B(\mathbf{p}''',\mathbf{p}'',\xi_2)\Lambda_\mathbf{q}^\lambda(\mathbf{p}',\mathbf{p}'',\omega)\delta(\xi_1-\xi_2-\omega)\\ =\frac{\pi}{\Omega}\sum_{\mathbf{p}'s}\sum_{\lambda=1}^{2}\Lambda_\mathbf{q}^\lambda(\mathbf{p}''',\mathbf{p},0)A(\mathbf{p}+\mathbf{q},\,\mathbf{p}'+\mathbf{q},\,\mu)\\ \times B(\mathbf{p}''',\mathbf{p}'',\mu)\Lambda_\mathbf{q}^\lambda(\mathbf{p}',\mathbf{p}'',0).\qquad(2.4)\end{aligned}$$

In this expression A and B are the spectral densities for the quasi-particle and quasi-hole propagators described in L-I and μ is the exact chemical potential.

III. FREE-ELECTRON GAS

We consider first the case of noninteracting electrons for which the only diagram occuring is that shown in Fig. 2. The contribution of this graph to $F(q,\omega)$ is

$$\begin{aligned}F_0(q,\omega)=\lim_{\eta\to0^+}\frac{e^2}{2\pi m^2}\sum_{\mathbf{p},\mathbf{p}'}\mathbf{p}_t\cdot\mathbf{p}_t'\\ \times\int_{-\infty}^{\infty}e^{i\omega t-\eta|t|}T\langle0|a_\mathbf{p}^\dagger(t)a_{\mathbf{p}+\mathbf{q}}(t)\\ \times a_{\mathbf{p}'+\mathbf{q}}^\dagger(0)a_{\mathbf{p}'}(0)|0\rangle dt\\ =\frac{e^2}{m^2}\sum_\mathbf{p}p_t^2|\langle0|a_\mathbf{p}^\dagger a_{\mathbf{p}+\mathbf{q}}|\mathbf{p}\rangle|^2\\ \times\int_0^\infty d\nu\,\delta(\epsilon_{\mathbf{p}+\mathbf{q}}-\epsilon_\mathbf{p}-\nu)\left(\frac{1}{\nu-\omega-i\eta}+\frac{1}{\nu+\omega-i\eta}\right),\end{aligned}$$

here $|0\rangle$ refers to the ground state of the noninteracting system.[11] $F_0(q,\omega)$ has a branch cut along the real ω axis across which the discontinuity is

$$\rho_0(q,\omega)=\frac{e^2}{m^2}\sum_{\substack{p\le k_F\\|\mathbf{p}+\mathbf{q}|>k_F}}p_t^2\delta(\epsilon_{\mathbf{p}+\mathbf{q}}-\epsilon_\mathbf{p}-\omega).$$

[10] N. Tzoar and A. Klein, Phys. Rev. **124**, 1297 (1961).

[11] $\epsilon_k=k^2/2m$ and the a's are fermion creation operators. Spin will be included by a factor of 2 in evaluating the momentum sums.

The restriction on the sum comes from the matrix element $\langle 0|a_{\mathbf{p}}{}^{\dagger}a_{\mathbf{p}+\mathbf{q}}|\mathbf{p}\rangle$. This expression is evaluated in Appendix A and we find in the anomalous limit, $\omega \to 0$. $q \ll k_F$,

$$\sigma(q) = \tfrac{3}{4}\pi(ne^2/m)(1/qv_F).$$

A crucial point in the formulation of quasi-particle theories is the specification of the current carried by a quasi-particle when excited by an external field. In the absence of impurities, quasi-particle propagation is described by a Green's function whose spectral representation is

$$G(\mathbf{k},\nu) = \lim_{\eta\to 0^+}\left(\int_{\mu}^{\infty}\frac{A(\mathbf{k},x)dx}{x-\nu-i\eta}+\int_{-\infty}^{\mu}\frac{B(\mathbf{k},x)dx}{x-\nu+i\eta}\right),$$

where $A(\mathbf{k},\mathbf{k}',x) = A(\mathbf{k},x)\delta_{\mathbf{k}\mathbf{k}'}$, $B(\mathbf{k},\mathbf{k}',x) = B(\mathbf{k},x)\delta_{\mathbf{k}\mathbf{k}'}$. In the present case, the transverse current measuring operator is

$$\mathbf{J}_{\mathbf{q}} = (e/m)\textstyle\sum_{\mathbf{k}}\mathbf{k}_t a_{\mathbf{k}+\mathbf{q}}{}^{\dagger}a_{\mathbf{k}}$$

(we consider only terms to second order in q/k_F). We may insert this into a hole line, e.g., by writing

$$\mathbf{L}_{\mathbf{q}}(\mathbf{k},t,t') = T\langle\Psi_0|a_{\mathbf{k}+\mathbf{q}}{}^{\dagger}(t)\mathbf{J}_q(t')a_{\mathbf{k}}(0)|\Psi_0\rangle.$$

The Fourier transform of this is

$$\mathbf{L}_{\mathbf{q}}(\mathbf{k},\nu,\omega) = G(\mathbf{k}+\mathbf{q},\,\nu+\omega)\mathbf{\Lambda}_{\mathbf{q}}(\mathbf{k},\omega)G(\mathbf{k},\nu),$$

where, in the absence of impurities,

$$\mathbf{\Lambda}_{\mathbf{q}}(\mathbf{k},\mathbf{k}',\omega) = \mathbf{\Lambda}_{\mathbf{q}}(\mathbf{k},\omega)\delta_{\mathbf{k}\mathbf{k}'}.$$

$\mathbf{L}_{\mathbf{q}}(\mathbf{k},\nu,\omega)$ is represented by the diagram of Fig. 3 where the shaded circle corresponds to $\mathbf{\Lambda}_{\mathbf{q}}(\mathbf{k},\omega)$. As long as $q\neq 0$, these quantities are continuous at $\omega=0$ and there is no difficulty in taking the $\omega\to 0$ limit. We easily obtain, e.g., for noninteracting electrons, $\mathbf{\Lambda}_{\mathbf{q}}(\mathbf{k},0) = (e/m)\mathbf{k}_t$.

It is now possible to derive a Ward identity for the vertex function. In the case of noninteracting electrons, we have simply

$$G_0(\mathbf{k},\nu) = \frac{1}{\epsilon_{\mathbf{k}}-\nu-i\alpha_{\mathbf{k}}},\qquad \begin{aligned}\alpha_{\mathbf{k}} &\to 0^+, & k&>k_F,\\ &\to 0^-, & k&\leq k_F.\end{aligned}$$

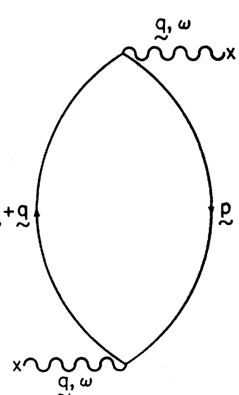

FIG. 2. The lowest order reduced graph, associated with noninteracting electrons.

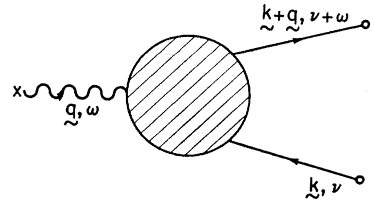

FIG. 3. Diagram representing $\mathbf{L}_{\mathbf{q}}(\mathbf{k},\nu,\omega)$.

To second order in q/k, for k near k_F, we find

$$e[G_0(\mathbf{k}+\mathbf{q},\,\nu)-G_0(\mathbf{k},\nu)]$$

$$= e\mathbf{q}\cdot\nabla_{\mathbf{k}}G_0(\mathbf{k},\nu) = \mathbf{q}\cdot\left(\frac{(e/m)(\mathbf{k}+\frac{1}{2}\mathbf{q})}{(\epsilon_{\mathbf{k}+\mathbf{q}}-\nu-i\alpha_{\mathbf{k}+\mathbf{q}})(\epsilon_{\mathbf{k}}-\nu-i\alpha_{\mathbf{k}})}\right).$$

That is,

$$e\nabla_{\mathbf{k}}G_0(\mathbf{k},\nu) \cong G_0(\mathbf{k}+\mathbf{q},\,\nu)\frac{e}{m}(\mathbf{k}+\tfrac{1}{2}\mathbf{q})G_0(\mathbf{k},\nu).$$

Finally, taking components transverse to $\mathbf{q}$,

$$e\nabla_{\mathbf{k}}G_0(\mathbf{k},\nu)|_t \cong G_0(\mathbf{k}+\mathbf{q},\,\nu)\mathbf{\Lambda}_q{}^{(0)}(\mathbf{k},0)G_0(\mathbf{k},\nu).$$

Consequently, inserting a current-measuring operator into a free-electron line is the same as differentiating with respect to the momentum. The argument following Eq. (3.28) of L-III can now be applied and leads to the relation

$$\mathbf{\Lambda}_{\mathbf{q}}(\mathbf{k},0) \cong \frac{e}{m}\mathbf{k} - e\nabla_{\mathbf{k}}\Sigma(\mathbf{k},E_{\mathbf{k}}),$$

where $E_{\mathbf{k}}$ is the quasi-particle energy and Σ is the self-energy function. By differentiating the equation[12]

$$\epsilon_{\mathbf{k}} - E_{\mathbf{k}} - \Sigma(\mathbf{k},E_{\mathbf{k}}) = 0, \tag{3.1}$$

we find

$$\mathbf{\Lambda}_{\mathbf{q}}(\mathbf{k},0) = e\mathbf{U}_t(\mathbf{k})/N_{\mathbf{k}}, \tag{3.2}$$

where $N_{\mathbf{k}} = (1+\partial\Sigma/\partial E_{\mathbf{k}})^{-1}$ and $\mathbf{U}(\mathbf{k})$ is the quasi-particle group velocity.

In the free-electron case, (2.4) now becomes

$$\sigma(q) = \frac{\pi e^2}{\Omega}\sum_{\substack{p\leq k_F\\ |\mathbf{p}+\mathbf{q}|>k_F}}\frac{|\mathbf{U}_t(\mathbf{p})|^2}{N_{\mathbf{p}}{}^2}\delta(E_{\mathbf{p}+\mathbf{q}}-\mu)\delta(E_{\mathbf{p}}-\mu). \tag{3.3}$$

In deriving this, we have made use of Luttinger's[12] result:

$$A(\mathbf{k},\mu) = \delta(E_{\mathbf{k}}-\mu)[1-\theta(k/k_F)],$$
$$B(\mathbf{k},\mu) = \delta(E_{\mathbf{k}}-\mu)\theta(k/k_F),$$

where

$$\theta(x) = 1 \quad \text{for} \quad x\leq 1,$$
$$\quad = 0 \quad \text{for} \quad x>1.$$

However, we find from (3.1) that

$$\mathbf{U}(\mathbf{p}) = \frac{p}{m} - \nabla_{\mathbf{p}}\Sigma(\mathbf{p},E_{\mathbf{p}}) - \mathbf{U}(\mathbf{p})\frac{\partial}{\partial E_{\mathbf{p}}}\Sigma(\mathbf{p},E_{\mathbf{p}}),$$

[12] J. M. Luttinger, Phys. Rev. **121**, 942 (1961).

so

$$\frac{\mathbf{U}(\mathbf{p})}{N_\mathbf{p}}=\nabla_\mathbf{p}[\epsilon_\mathbf{p}-\Sigma(\mathbf{p},E_\mathbf{p})],$$

and

$$\delta(E_\mathbf{p}-\mu)=\frac{\delta(p-k_F)}{|\nabla_\mathbf{p}[\epsilon_\mathbf{p}-\Sigma(\mathbf{p},\mu)]|}.$$

Changing the sum in (3.3) to an integral and taking the polar axis of a spherical coordinate system along $\mathbf{q}$, we find

$$\begin{aligned}\sigma(q)&=\frac{\pi e^2}{(2\pi)^3}\int p^3dp\int d\varphi\int \sin^3\theta d\theta\,\frac{|\mathbf{U}(\mathbf{p})|^2}{N_\mathbf{p}}\delta(p-k_F)\\&\quad\times\delta\{E_\mathbf{p}-\Sigma(\mathbf{p},\mu)-\mu-q|\nabla_\mathbf{p}[E_\mathbf{p}-\Sigma(\mathbf{p},\mu)]|\cos\theta\}\\&=\frac{\pi e^2k_F{}^2}{(2\pi)^3}\frac{U_F}{N_F}\int dx\,\delta\left(q\frac{U_F}{N_F}x\right)(1-x^2)\\&=\frac{\pi e^2k_F{}^3}{m(2\pi)^2qv_F}=\frac{3\pi}{4}\frac{ne^2}{m}\frac{1}{qv_F},\end{aligned}\tag{3.4}$$

where $x=\cos\theta$.

U_F and N_F are the magnitude of the group velocity and wave function renormalization constant, respectively, at the Fermi surface, and $v_F=k_F/m$. We have again made free use of Luttinger's[13] results concerning the Fermi surface of the free-electron gas. Equation (3.4) is precisely the result obtained for noninteracting electrons in the anomalous limit since v_F is the same for interacting and noninteracting electrons. This means that for small q the Fourier transform of current distribution associated with a quasi-particle of momentum $\mathbf{k}$ has the form

$$j_\mathbf{k}(\mathbf{q})=\frac{e\mathbf{k}}{m}-\frac{e}{m}\frac{(\mathbf{k}\cdot\mathbf{q})\mathbf{q}}{q^2}.$$

The second term on the right must come from backflow associated with the screening cloud and thus for a transverse field, $\mathbf{k}\cdot\mathbf{q}=0$ and the particle behaves as if unscreened.

IV. IMPURITY EFFECTS

The procedure for including impurity interaction is described in detail in L-I. Using the same technique here, we shall calculate with the impurity potential

$$V_{\text{imp}}=\sum_{s=1}^{N_i}v(\mathbf{r}-\mathbf{r}_s),$$

and then average over random configurations of these impurities. For any configuration S the real conductivity may be written in the form (2.4). This must be averaged over all configurations S. In computing this average we retain terms to only first order in the density of impurities n_i. Since each impurity interaction which appears in a graph contributes a factor of n_i, we may neglect such effects in the vertex parts, since they would then enter as $n_i{}^2$, and we replace the vertex function by

$$\mathbf{\Lambda}_\mathbf{q}(\mathbf{p},\mathbf{p}',0)=\frac{e}{m}g(p)\mathbf{p}_t\delta_{\mathbf{p}\mathbf{p}'},$$

where $g(p)=m|\mathbf{U}(\mathbf{p})|/pN_\mathbf{p}$. Then Eq. (2.4) may be written

$$\begin{aligned}\sigma(q)=\frac{e^2\pi}{m^2\Omega}\sum_{\mathbf{p},\mathbf{p}'}g(p)g(p')\\ \times\mathbf{p}_t\cdot\mathbf{p}_t'\langle A(\mathbf{p}+\mathbf{q},\mathbf{p}'+\mathbf{q},\mu)B(\mathbf{p}',\mathbf{p},\mu)\rangle_{\text{av}},\end{aligned}$$

where the averaging is as described in L-I. Since the spectral density functions occuring here are just the discontinuities of the propagators across the real axis, we have

$$\begin{aligned}&\langle A(\mathbf{p}+\mathbf{q},\mathbf{p}'+\mathbf{q},\mu)B(\mathbf{p}',\mathbf{p},\mu)\rangle_{\text{av}}\\&=\langle[S(\mathbf{p}+\mathbf{q},\mathbf{p}'+\mathbf{q};\mu+i\epsilon)-S(\mathbf{p}+\mathbf{q},\mathbf{p}'+\mathbf{q};\mu-i\epsilon)]\\&\qquad\times[S(\mathbf{p}',\mathbf{p};\mu+i\epsilon)-S(\mathbf{p}',\mathbf{p},\mu-i\epsilon)]\rangle_{\text{av}}.\end{aligned}$$

Therefore we have need only to calculate expressions of the form

$$K_\mathbf{q}(\mathbf{p};\pm\pm)=\textstyle\sum_{\mathbf{p}'}g(p')\mathbf{p}_t\cdot\mathbf{p}_t'F_\mathbf{q}(\mathbf{p}',\mathbf{p},\pm\pm),$$

where

$$F_\mathbf{q}(\mathbf{p}',\mathbf{p},\pm\pm)=\langle S(\mathbf{p}'+\mathbf{q},\mathbf{p}+\mathbf{q},\mu\pm i\epsilon)S(\mathbf{p},\mathbf{p}',\mu\pm i\epsilon)\rangle_{\text{av}}.$$

It follows from the analytic structure of the propagators that

$$S(\mathbf{p},\mathbf{p}',\omega)=-S^*(\mathbf{p}',\mathbf{p},\omega^*),$$

so

$$\begin{aligned}F_\mathbf{q}(\mathbf{p}',\mathbf{p},--)&=F_\mathbf{q}{}^*(\mathbf{p},\mathbf{p}',++),\\F_\mathbf{q}(\mathbf{p}',\mathbf{p},-+)&=F_\mathbf{q}{}^*(\mathbf{p},\mathbf{p}',+-).\end{aligned}$$

Consequently, the conductivity may be written

$$\sigma(q)=\frac{2e^2\pi}{m^2\Omega}\sum_\mathbf{p}g(p)\,\mathrm{Re}[K_\mathbf{q}(\mathbf{p},++)-K_\mathbf{q}(\mathbf{p},+-)].\tag{4.1}$$

Because of time-reversal invariance,

$$S(\mathbf{p}',\mathbf{p},\omega)=S(-\mathbf{p},-\mathbf{p}',\omega),$$

and

$$\begin{aligned}&F_\mathbf{q}(\mathbf{p}',\mathbf{p},\pm\pm)\\&\quad=\langle S(\mathbf{p}'+\mathbf{q},\mathbf{p}+\mathbf{q},\mu\pm i\epsilon)S(-\mathbf{p}',-\mathbf{p},\mu\pm i\epsilon)\rangle_{\text{av}}.\end{aligned}$$

In the graphical representation of the product of two propagators, there occur terms such as shown in Fig. 4(a). When the average is taken, it happens that impurity interaction parts on the two electron lines become joined and although total momentum is conserved, there can be a net momentum transfer between the electrons. Following the arguments in L-I, we are lead to the

[13] J. M. Luttinger, Phys. Rev. **119**, 1153 (1960).

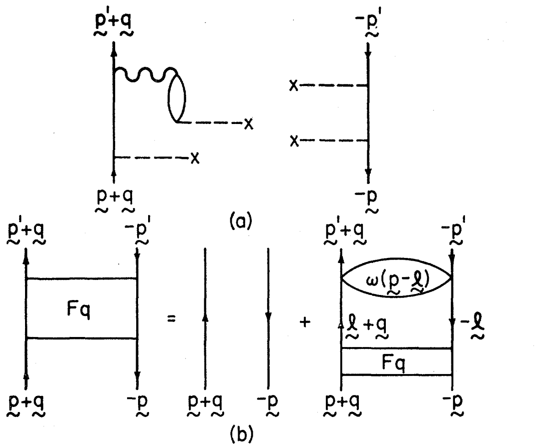

FIG. 4. (a) A contribution to the product of two propagators for a given impurity distribution. (b) Diagramatical representation of Eq. (4.2).

integral equation

$$F_{\mathbf{q}}(\mathbf{p}', \mathbf{p}, \pm\pm) = \bar{S}(\mathbf{p}+\mathbf{q}, \pm)\bar{S}(\mathbf{p}, \pm) \times\left\{\delta_{\mathbf{pp}'}-\frac{1}{\Omega}\sum_{l} W(\mathbf{p}-\mathbf{l}, \pm\pm)F_{\mathbf{q}}(\mathbf{l}, \mathbf{p}', \pm\pm)\right\}. \quad (4.2)$$

$\bar{S}$ is the averaged propagator discussed in L-I and $W(\mathbf{l}, \pm\pm)$ is the sum of all irreducible interaction parts, which depend only on the momentum exchanged between the two particle lines. This equation is shown schematically in Fig. 4(b). When $\mathbf{q}=0$, Eq. (4.2) differs from Eq. (6.9) of L-I only in that the factor $\mathbf{p}\cdot\mathbf{p}'$ which appears there is replaced by $\mathbf{p}_t\cdot\mathbf{p}_t'$. The dependence on $\mathbf{q}$ rules out a solution along the lines of that in L-I in general, but we can distinguish two cases. The anomalous limit corresponds to $q\ll k_f$, $qk_F/m\Gamma_F\gg1$, and the normal limit to $qk_F/m\Gamma_F\ll1$. Here, Γ_F is the imaginary part of the self-energy of an electron at the Fermi surface. In the latter case an expansion in powers of qv_F/Γ_F is possible. This is carried out in Appendix B. We now consider the anomalous limit where such an expansion is not possible.

The argument in L-I following Eq. (8.10) can be applied also in our case to show that, since the dominant contribution to the conductivity comes from the vicinity of the Fermi surface, we need retain only the term $F_{\mathbf{q}}(\mathbf{p}', \mathbf{p}, +-)$ in (4.1). This is discussed further in Appendix B. We, therefore, drop the frequency indices and write simply,

$$\sigma(q) = -(2\pi e^2/m^2\Omega)\textstyle\sum_{\mathbf{p}} g(p)\ \mathrm{Re}K_{\mathbf{q}}(\mathbf{p}). \quad (4.3)$$

Making use of the approximations, valid in the anomalous limit,

$$\bar{S}(\mathbf{p}+\mathbf{q}, +)\bar{S}(\mathbf{p}, -)\cong-\frac{1}{4\pi}\frac{\delta(e_p)}{\Gamma_F-i\mathbf{q}\cdot\mathbf{U}(k_F)/N_F},$$

and

$$\delta(e_p)\cong(N_Fk_F/mU_F)\delta(\epsilon_p-\mu),$$

the latter following directly from L-I Eq. (8.8), we find from (4.2)

$$\left[\Gamma_F-i\frac{\mathbf{q}\cdot\mathbf{U}(k_F)}{N_F}\right]F_{\mathbf{q}}(\mathbf{p}',\mathbf{p}) = -\frac{1}{4\pi}\delta(e_p)\left(\delta_{\mathbf{p},\mathbf{p}'}-\frac{1}{\Omega}\sum_{\mathbf{l}} W(\mathbf{p}-\mathbf{l})F_{\mathbf{q}}(\mathbf{p}',\mathbf{l})\right). \quad (4.4)$$

In terms of the quantity

$$f_{\mathbf{q}}(\mathbf{p}) = -(4\pi e/m)\textstyle\sum_{\mathbf{p}'} F_{\mathbf{q}}(\mathbf{p}',\mathbf{p})[\mathbf{p}_t'\cdot\mathbf{E}(\mathbf{q})]g(p'),$$

where E is the external field, (4.4) reduces to the equation

$$\left[\Gamma_F-\frac{i\mathbf{q}\cdot\mathbf{U}(k_F)}{N_F}\right]f_{\mathbf{q}}(\mathbf{p}) = \left\{e\sum_{\mathbf{p}'}\frac{\mathbf{p}_t'\cdot\mathbf{E}}{m}g(p')\delta_{\mathbf{p},\mathbf{p}'} -\frac{e}{\Omega}\sum_{\mathbf{p},\mathbf{l}} W(\mathbf{p}',\mathbf{l})F_{\mathbf{q}}(\mathbf{p}',l)g(\mathbf{p}')\left[\frac{\mathbf{p}_t'\cdot\mathbf{E}(\mathbf{q})}{m}\right]\right\}\delta(e_p).$$

This may be written, defining $\tilde{\Gamma}_{\mathbf{k}}=N_{\mathbf{k}}\Gamma_{\mathbf{k}}$,

$$[\tilde{\Gamma}_F-i\mathbf{q}\cdot\mathbf{U}(k_F)]f_{\mathbf{q}}(\mathbf{p}) = e\{\mathbf{U}_t(k_F)\cdot\mathbf{E}(\mathbf{q})\}\frac{\partial f_0}{\partial E_{\mathbf{p}}} +\frac{N_F}{4\pi\Omega}\sum_{\mathbf{l}} W(\mathbf{p}-\mathbf{l})f_q(\mathbf{l})\delta(e_p), \quad (4.5)$$

where f_0 is the zero-temperature Fermi distribution with chemical potential μ. If $P_{\mathbf{l}\mathbf{l}'}$ is the scattering probability for quasi-particle states (which must be normalized to one quasi-particle per unit volume, see L-III), then

$$\tilde{\Gamma}_F=\sum_{\mathbf{l}} P_{\mathbf{p}\mathbf{l}}\delta(e_p),$$

$$\sum_{\mathbf{l}} P_{\mathbf{l}\mathbf{p}}f_{\mathbf{q}}(l)=\frac{N_F}{4\pi\Omega}\sum_{\mathbf{l}} W(\mathbf{p}-\mathbf{l})f_{\mathbf{q}}(\mathbf{l})\delta(e_p),$$

and (4.5) becomes

$$-i\mathbf{q}\cdot\mathbf{U}(k_F)f_{\mathbf{q}}(p)+e[\mathbf{U}_t(\mathbf{p})\cdot\mathbf{E}(\mathbf{q})]\frac{\partial f_0}{\partial E_{\mathbf{p}}} =\sum_{\mathbf{l}} P_{\mathbf{p}\mathbf{l}}\{f_{\mathbf{q}}(\mathbf{l})-f_{\mathbf{q}}(\mathbf{p})\}=\frac{f_q(\mathbf{p})}{\tau}. \quad (4.6)$$

Equation (4.6) is the low-frequency form of the equation derived by Silin from Landau's theory. In particular, for noninteracting electrons this reduces precisely to the Boltzmann equation. We have therefore shown that the latter is valid if the condition $1/\tau \ll \mu$ is satisfied. This point has also been made by Chester.[14]

ACKNOWLEDGMENTS

The author wishes to thank Dr. J. S. Langer and Professor J. M. Radcliffe for many helpful discussions and suggestions.

APPENDIX A

We first replace the summation by an integral. In the case that $q>2k_F$ the condition $|\mathbf{q}+\mathbf{p}|>k_F$ is satisfied by all momenta inside the Fermi sphere so

$$S=\sum p_t^2\delta(\epsilon_{\mathbf{p}+\mathbf{q}}-\epsilon_{\mathbf{p}}-\omega)=\frac{\Omega}{(2\pi)^3}\int_0^{k_F}p^4dp\int_0^{2\pi}d\varphi \times\int_0^{\pi}\sin^3\theta d\theta\,\delta(\epsilon_{\mathbf{p}+\mathbf{q}}-\epsilon_{\mathbf{p}}-\omega),$$

where we have taken the polar axis along $\mathbf{q}$ so $p_t^2 = p^2\sin^2\theta$.

$$\epsilon_{\mathbf{p}+\mathbf{q}}-\epsilon_{\mathbf{p}}=\epsilon_{\mathbf{q}}+(pq/m)\cos\theta,$$

so, from the δ-function condition, we find $\cos\theta = m(\omega-\epsilon_q)/p_q$. Let

$$x=\cos\theta,\quad x_0=m(\omega-\epsilon_q)/p_q;$$

then

$$S=\frac{m\Omega}{4\pi^2 q}\int_0^{k_F}p^3dp\int_{-1}^{1}dx(1-x^2)\delta(x-x_0).$$

To get a nonzero value we must have $0\leq x_0^2\leq 1$ or $0\leq (m^2/p^2q^2)(\omega-\epsilon_q)^2\leq 1$. If $k_0^2=m^2(\omega-\epsilon_q)^2/q^2$, then the range of the p integration is given by $0\leq k_0^2\leq p^2\leq k_F^2$. Hence,

$$S=\frac{m}{4\pi^2 q}\int_{k_0}^{k_F}p^3dp\int_{-1}^{1}dx(1-x^2)\delta(x-x_0) =\frac{m}{16\pi^2 q}(k_F^2-k_0^2).\quad\text{(A1)}$$

When $q<2k_F$, there is the additional restriction $\epsilon_{\mathbf{p}+\mathbf{q}}>\epsilon_F$. This implies that $\epsilon_{\mathbf{p}}>\epsilon_F-\omega$ so we have the two conditions for a nonzero result

$$0<k_0=m|\omega-\epsilon_q|/q\leq p\leq k_F,\quad p>(2m|\epsilon_F-\omega|)^{1/2}.$$

In the case $k_0<(2m|\epsilon_F-\omega|)^{1/2}$, the lower limit of the p integral in (A1) must be replaced by $[2m(\epsilon_F-\omega)]^{1/2}$. In the case $k_0>(2m|\epsilon_F-\omega|)^{1/2}$, (A1) is still valid. Thus, we obtain

$$\mathrm{Re}\sigma(q,\omega)=0\quad\text{for}\quad \omega<\epsilon_q-\frac{qk_F}{m},\quad \omega>\epsilon_q+\frac{qk_F}{m},$$

$$=\frac{3\pi}{4}\frac{ne^2}{m}\frac{1}{qv_F}\left[1-\frac{q^2}{4k_F^2}-\frac{\omega^2}{q^2v_F^2}\right]\quad\text{for}\quad \omega<-\frac{\epsilon_Fq^2}{k_F^2}+\frac{2\epsilon_Fq}{k_F},$$

$$=\frac{3\pi k_F}{16q\omega}\frac{ne^2}{m}\left[1-\left(\frac{\omega}{qv_F}-\frac{q}{2k_F}\right)^2\right]^2\quad\text{for}\quad \omega<-\frac{\epsilon_Fq^2}{k_F^2}+\frac{2\epsilon_Fq}{k_F}.$$

This is precisely the result obtained by Lindhard.[15]

APPENDIX B

We wish to discuss here the solution of Eq. (4.2) in the limit of the normal skin effect, $qk_F/m\Gamma\ll 1$. In this limit, it is possible to expand $K_{\mathbf{q}}(\mathbf{p},\pm\pm)$ to first order in q, e.g.,

$$K_{\mathbf{q}}(\mathbf{p},+-)\cong K(\mathbf{p},+-)+\mathbf{q}\cdot\nabla_{\mathbf{q}}[K_{\mathbf{q}}(\mathbf{p},+-)]_{q=0}.$$

Now $K(\mathbf{p},+-)$ is just the quantity $K_{+-}(\mathbf{p})$ considered in L-I except that the transverse, rather than the longitudinal, component of the momenta occurs. Since the spherical average of p^2 and p_t^2 differ only by a factor of 2/3, we find

$$K(\mathbf{p},+-)\cong-\frac{2}{3}\frac{k_F^2N_Fg(k_F)}{4\pi U_F(\Gamma_F-\Gamma_F')}\delta(p-k_F).\quad\text{(B1)}$$

Let us define $\mathbf{G}(\mathbf{p})=\nabla_{\mathbf{q}}K(\mathbf{p},+-)|_{q=0}$ and derive an integral equation for this quantity. Dropping the frequency indices in (4.2), we find

$$\nabla_{\mathbf{q}}F_{\mathbf{q}}(\mathbf{p}',\mathbf{p})=\nabla_{\mathbf{q}}[\bar{S}(\mathbf{p}+\mathbf{q})\bar{S}(\mathbf{p})]\left(\delta_{\mathbf{p},\mathbf{p}'}-\frac{1}{\Omega}\sum_{\mathbf{l}}W(\mathbf{p}-\mathbf{l})F_{\mathbf{q}}(\mathbf{l},\mathbf{p}')\right)-\bar{S}(\mathbf{p}+\mathbf{q})\bar{S}(\mathbf{p})\left(\frac{1}{\Omega}\sum_{\mathbf{l}}W(\mathbf{p}-\mathbf{l})\nabla_{\mathbf{q}}F_{\mathbf{q}}(\mathbf{l},\mathbf{p}')\right).$$

In terms of the quantities

$$Q_\pm(p)=\epsilon_p-\mu\pm i\epsilon-\Delta(p,\mu)\pm i\Gamma(p,\mu),$$

we find

$$\mathbf{G}(p)=-\frac{1}{4\pi^2}\frac{\nabla_{\mathbf{p}}Q_-(p)}{Q_+(p)Q^2(p)}\left(g(p)p_t^2-\frac{1}{\Omega}\sum_{\mathbf{l}}\sum_{\mathbf{p}'}g(p')(\mathbf{p}_t\cdot\mathbf{p}_t')W(\mathbf{l}-\mathbf{p})F_0(\mathbf{l},\mathbf{p})\right) -\bar{S}(\mathbf{p})\bar{S}(\mathbf{p})\frac{1}{\Omega}\sum_{\mathbf{l}}\sum_{\mathbf{p}'}g(p')\mathbf{p}_t\cdot\mathbf{p}_t'W(\mathbf{l}-\mathbf{p})\nabla_{\mathbf{q}}F(\mathbf{l},\mathbf{p}')_{q=0}.\quad\text{(B2)}$$

[14] G. V. Chester (unpublished).

[15] J. Lindhard, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. 28, 8 (1954).

The quantity $W(\mathbf{l}-\mathbf{p})$ depends only on p, l, and $\theta_{p,l}$; $F_0(\mathbf{l},\mathbf{p}')$ and $\nabla_{\mathbf{q}}F_{\mathbf{q}}(\mathbf{l},\mathbf{p}')_{q=0}$ depend only on l, p', and $\theta_{lp'}$. Note that to second order in q/p we may replace $\mathbf{p}_t\cdot\mathbf{p}_t'$ by $(\mathbf{p}_t\cdot\mathbf{l})(\mathbf{p}_t'\cdot\mathbf{l})/l^2$ since in a coordinate system with l as the polar axis,

$$\cos\theta_{p_tp_t'}=\cos\theta_{lp_t'}\cos\theta_{lp_t}+\sin\theta_{lp_t}\sin\theta_{lp_t'}\cos\varphi,$$

and the second term vanishes when the φ part of the p' integration is carried out. Hence, (B2) may be put in the form

$$\mathbf{G}(p)=-\frac{1}{4\pi^2}\frac{1}{Q_+(p)Q_-(p)}\left\{\frac{\nabla_{\mathbf{p}}Q_-(p)}{Q_-(p)}\left[g(p)p_t^2-\frac{1}{\Omega}\sum_{\mathbf{l}}\sum_{\mathbf{p}'}g(p')\frac{(\mathbf{p}_t\cdot\mathbf{l})(\mathbf{p}_t'\cdot\mathbf{l})}{l^2}W(\mathbf{l}-\mathbf{p})F_0(\mathbf{l},\mathbf{p}')\right]\right.$$
$$\left.-\frac{1}{\Omega}\sum_{\mathbf{l},\mathbf{p}'}g(p')\frac{(\mathbf{p}_t\cdot\mathbf{l})(\mathbf{p}_t'\cdot\mathbf{l})}{l^2}W(\mathbf{l}-\mathbf{p})\nabla_{\mathbf{q}}F_0(\mathbf{l},\mathbf{p}')\right\}.$$

Letting

$$W(l,p)=\frac{1}{4\pi^2}\int_0^\pi W(\mathbf{l}-\mathbf{p})\cos\theta\sin\theta d\theta_{lp},$$

we find to order q^2

$$\mathbf{G}(p)=\frac{1}{4\pi^2}\frac{1}{Q_+(p)Q_-(p)}\left\{\frac{\nabla_{\mathbf{p}}Q_-(p)}{Q_-(p)}\left[g(p)p_t^2-p_t\int_0^\infty lW(l,p_t)K(\mathbf{l}+-)dl\right]+p_t\int_0^\infty lW(l,p_t)\mathbf{G}(l)dl\right\}.$$

Inserting the value of $K(\mathbf{l}+-)$ from (B1), we find the equation

$$\mathbf{G}(p)=-\frac{1}{4\pi^2}\frac{1}{Q_+(p)Q_-(p)}\left\{\frac{\nabla_{\mathbf{p}}Q_-(p)}{Q_-(p)}\left[g(p)p_t^2+\frac{p_tk_F^3g(k_F)N_F}{4\pi U_F}\frac{W(k_F,p_t)}{\Gamma_F-\Gamma_F'}\right]+p_t\int_0^\infty lW(l,p_t)\mathbf{G}(l)dl\right\}.$$

The expression in front of the curly brackets is sharply peaked near the Fermi surface, hence we need consider only values of p near k_F. Also, to order q^2 we may replace p_t^2 by p^2 if we correct the result by a factor of 2/3. Then, in terms of quantities

$$\lambda=\frac{\nabla_{\mathbf{p}}Q_-(p)}{Q_-(p)},\quad \alpha=\frac{2k_F^3N_Fg(k_F)}{12\pi U_F(\Gamma_F-\Gamma_F')},\quad \mathbf{M}=\int_0^\infty lW(k_F,l)\mathbf{G}(l)dl,$$

we find

$$\mathbf{G}(p)=-\frac{1}{4\pi^2}\frac{1}{Q_+(p)Q_-(p)}\{\lambda[g(p)p^2+p\alpha w(p,k_F)]+p\mathbf{M}\}. \tag{B3}$$

In L-I, Eq. (8.8), it is pointed out that we may make the replacement

$$\frac{1}{4\pi^2}\frac{w(p,k_F)}{Q_+(p)Q_-(p)}\cong\frac{\Gamma_F'}{k_F^2\Gamma_F}\delta(p-k_F),$$

if it occurs in a convergent integral with a slowly varying function of p. Anticipating this and integrating both sides of (B3) over p we obtain an equation for M:

$$\mathbf{M}=-\frac{\Gamma_F'}{\Gamma_F}\{\lambda[g(k_F)k_F+\alpha w(k_F,k_F)]+\mathbf{M}\}.$$

Thus, we find

$$\mathbf{M}=-\Gamma_F'\lambda[g(k_F)k_F+\alpha w(k_F,k_F)]/(\Gamma_F-\Gamma_F').$$

Replacing this in (B3), we obtain for p near k_F

$$\nabla_{\mathbf{q}}K_{\mathbf{q}}(p)|_{q=0}\cong-\frac{N_Fk_F}{4\pi U_F}\lambda\frac{[g(k_F)+\alpha w(k_F,k_F)]}{\Gamma_F-\Gamma_F'}\delta(p-k_F).$$

The approximations indicated are valid to order Γ_F/μ and so are good to first order in the impurity density. Thus we have

$$K_{\mathbf{q}}(\mathbf{p},+-)=-\frac{2k_F{}^2N_Fg(k_F)}{12\pi U_F}\frac{\delta(p-k_F)}{\Gamma_F-\Gamma_F{}'}g(k_F)\left[1+i\frac{\mathbf{q}\cdot\nabla_{\mathbf{p}}Q_-(k_F)}{\Gamma_F}\left(1+\frac{\Gamma_F{}'}{\Gamma_F-\Gamma_F{}'}\right)\right]$$
$$+O\left(\frac{qk_F}{m\Gamma_F}\right)^2=-\frac{k_F{}^2N_Fg(k_F)}{6\pi U_F}\frac{\delta(p-k_F)}{[\Gamma_F-\Gamma_F{}'-i\mathbf{q}\cdot\nabla_{\mathbf{p}}Q_-]_{p=k_F}}+O\left(\frac{qk_F}{m\Gamma_F}\right)^2.$$

Finally, since

$$\nabla_{\mathbf{p}}Q_-(k_F)=\nabla_{\mathbf{p}}[\epsilon_{\mathbf{p}}-\mu-\Sigma(\mathbf{p},\mu)]_{p=k_F}=\frac{\mathbf{U}_F}{N_F},$$

we obtain

$$K_{\mathbf{q}}(\mathbf{p},+-)\cong-\frac{k_F{}^2N_F{}^2g(k_F)}{6\pi U_F}\frac{\delta(p-k_F)}{\tilde{\Gamma}_F-\tilde{\Gamma}_F{}'-i\mathbf{q}\cdot\mathbf{U}_F}. \quad \text{(B4)}$$

We note that this is valid only for p near k_F, and it does not correspond to a solution of (B2) for values of p much larger than k_F, so it cannot be used, by itself, to evaluate $\sigma(q)$. Following the argument in L-I it is easily shown that $K_{\mathbf{q}}(\mathbf{p},++)$, while not contributing for p near k_F, produces a convergent integrand when combined with $K_{\mathbf{q}}(\mathbf{p},+-)$ for other values of p. Hence, we are justified in neglecting $K_{\mathbf{q}}(\mathbf{p},++)$ and using (B4) together with (4.5) to evaluate $\sigma(q)$. Inserting (B4) in (4.3), we obtain

$$\sigma(q)=\frac{2e^2\pi}{m^2\Omega}\frac{2}{3}\frac{k_F{}^2N_F{}^2g(k_F)}{4\pi U_F}\sum_{\mathbf{p}}\delta(p-k_F)\times\mathrm{Re}\{[\tilde{\Gamma}_F-\tilde{\Gamma}_F{}'-i\mathbf{q}\cdot\mathbf{U}_F(p)]^{-1}\}. \quad \text{(B5)}$$

It is pointed out in L-III that we should identify $\tilde{\Gamma}_F-\tilde{\Gamma}_F{}'$ with $1/2\tau$, where τ is the decay rate for a quasiparticle. Thus, after a simple integration (B5) reduces to

$$\sigma(q)=\frac{U_F\tau e^2k_F{}^2}{6\pi^2}\int_{-1}^{1}\frac{dx}{1+4q^2U_F{}^2\tau^2x^2},$$

where x is the cosine of the angle between $U(k_F)$ and q. In the limit $qv_F\tau\to 0$, we obtain

$$\sigma(q)=\frac{ne^2}{m}\tau\left(\frac{mU_F}{k_F}\right).$$

This is just the result obtained by Silin[1] in this limit.